The geometric and physical interpretation of fractional order derivatives of polynomial functions

M.H. Tavassoli, A. Tavassoli, M.R. Ostad Rahimi

Abstract. In this paper, after a brief mention of the definitions of fractional-order derivatives, we present a geometric interpretation of the tangent line angle of a polynomial with coefficients of fractional derivative. Then a comparison of the divergence of a gradient vector field in normal mode with the divergence of a vector field gradient fractions is performed. Finally, we show that there is a relationship between fractional derivative of polynomials at the tangent points and the order of the fractional derivative.

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1 Introduction

It is generally known that integer-order derivatives and integrals have clear physical and geometric interpretations, which significantly simplify their use for solving applied problems in various fields of science.

However, in case of fractional-order integration and differentiation, which represent a rapidly growing field both in theory and in applications to real-world problems, it is not so. Since the appearance of the idea of differentiation and integration of arbitrary (not necessary integer) order there was not any acceptable geometric and physical interpretation of these operations for more than 300 years [2, 10, 15, 5].

Fractional integration and fractional differentiation are generalizations of notions of integer-order integration and differentiation, and include n^{th} derivatives and n-fold integrals (n denotes an integer number) as particular cases. Because of this, it would be ideal to have such physical and geometric interpretations of fractional-order operators, which will provide also a link to known classical interpretations of integer-order differentiation and integration.

Obviously, there is still a lack of geometric and physical interpretation of fractional integration and differentiation, which is comparable with the simple interpretations of their integer-order counterparts.

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During the last two decades several authors have applied the fractional calculus in the field of sciences, engineering and mathematics (see [9, 4, 11, 6]). Mathematician Liouville, Riemann, and Caputo have done major work on fractional calculus, thus Fractional Calculus is a useful mathematical tool for applied sciences. Podlubny suggested a solution of more than 300 years old problem of geometric and physical interpretation of fractional integration and differentiation in 2002, for left-sided and right-sided of Riemann-Liouville fractional integrals [13, 1, 7].

J. A. Tenreiro Machado in 2003, presented a probabilistic interpretation of fractional order derivative, based on Grunwald –Letnikov definition of fractional order differentiation [8].

In this paper a new geometric interpretation for properties of polynomial's tangent line is defined as an area of a triangle and then the relationship between this area and order of differentiation is investigated.

Finally, it is shown that the area univocally increase or decrease according to the increasing of order of fractional derivative, except in the case where the order of derivative is equal to 0.5. some application of fractional derivatives in divergence of vector field gradient is also illustrated.

2 Definitions of fractional order derivatives

A number of researchers in this field have defined the fractional derivatives in different ways.[14, 12]

2.1 The Grunwald-Letnikov definition

$$ {}_{a}^{GL}D_{x}^{\alpha} f(x) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{n=0}^{\left[\frac{x-a}{h}\right]} (-1)^n \begin{pmatrix} \alpha \\ n \end{pmatrix} f(x-nh) \tag{2.1}$$

where $a = x - nh \Rightarrow n = \frac{x-a}{h}$.

2.2 The Riemann-Liouville definition

The Riemann-liouville derivative of order α and with lower limit a is defined as:

$$ {}_{a}^{RL}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{f(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau \tag{2.2}$$

where n is integer, α is real number and $(n-1) \leq \alpha < n$.

2.3 The M. Caputo (1967) definition

Caputo derivatives of order α are defined as:

$$ {}_{a}^{c}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau, \tag{2.3}$$

where n is integer, α is real number and $(n-1) \leq \alpha < n$.

3 Fractional derivatives properties

3.1 Definitions of Oldham and Spanier (1974)

The scaling property of fractional derivatives is described by:

$$\frac{d^\alpha f\left(\beta x\right)}{dx^\alpha} = \beta^\alpha \frac{d^\alpha f\left(\beta x\right)}{d(\beta x)^\alpha}. \tag{3.1}$$

This makes it suitable for the study of scaling and scale invariance. There is connection between local-scaling, box-dimension of an irregular function and order of fractional derivative.

3.2 Linearity

Fractional differentiation is a linear operation:

$$D^\alpha\left(\mu f(x) + \omega g(x)\right) = \mu D^\alpha f(x) + \omega D^\alpha g(x), \tag{3.2}$$

where D^α denotes any mutation of the fractional differentiation considered in this paper.

3.3 Definitions of K. S. Miller and B. Ross (1993)

$$\begin{gathered} D^\alpha f(x) = D^{\alpha_1} D^{\alpha_2} \ldots D^{\alpha_n} f(x) \\ \alpha = \alpha_1 + \alpha_{2+\cdots+}\alpha_n \\ \alpha_i < 1 \end{gathered} \tag{3.3}$$

This definition of sequential composition is very useful concept for obtaining fractional derivative of an arbitrary order. The derivative operator can be any definition Rimann-Liouville or Caputo.

4 Geometric and physical interpretation of fractional order derivatives

Geometrical and physical interpretations of integer order derivative and integral are defined in a simple way. The fractional order derivative and fractional order integral are not yet well established in simple way. In this paper, a simple interpretation of fractional order derivative is presented, which is useful in the applications of the subject.

The fractional order derivatives of a polynomial function can be computed by the formula

$$D^\alpha\left[x^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, \tag{4.1}$$

where α is the order of derivative and $0 < \alpha < 1$. By using the formula given in (4.1) and the property (3.2), the fractional derivative values of functions $f(x) = x^3$ and $g(x) = x^4 + x^3$ at $x = 2$ were computed and shown in table 4.1 and table 4.2 respectively.

Table 4.1

Fractional order derivatives	Fractional derivative values at $x = 2$ $m = \tan\theta$	$\theta = \tan^{-1}m$ (in radian)	Area of triangle ($\triangle$)
$\mathbf{D}^{0.1}[f(x)]$	$m_{0.1} = 8.4512$	$\theta_{0.1} = 1.4530$	$\triangle PA_{0.1}B = 3.7865$
$\mathbf{D}^{0.2}[f(x)]$	$m_{0.2} = 8.9018$	$\theta_{0.2} = 1.4589$	$\triangle PA_{0.2}B = 3.5948$
$\mathbf{D}^{0.3}[f(x)]$	$m_{0.3} = 9.3482$	$\theta_{0.3} = 1.4642$	$\triangle PA_{0.3}B = 3.4231$
$\mathbf{D}^{0.4}[f(x)]$	$m_{0.4} = 9.7866$	$\theta_{0.4} = 1.4690$	$\triangle PA_{0.4}B = 3.2698$
$\mathbf{D}^{0.5}[f(x)]$	$m_{0.5} = 10.2129$	$\theta_{0.5} = 1.4732$	$\triangle PA_{0.5}B = 3.1333$
$\mathbf{D}^{0.6}[f(x)]$	$m_{0.6} = 10.6226$	$\theta_{0.6} = 1.4769$	$\triangle PA_{0.6}B = 3.0124$
$\mathbf{D}^{0.7}[f(x)]$	$m_{0.7} = 11.0111$	$\theta_{0.7} = 1.4802$	$\triangle PA_{0.7}B = 2.9062$
$\mathbf{D}^{0.8}[f(x)]$	$m_{0.8} = 11.3734$	$\theta_{0.8} = 1.4831$	$\triangle PA_{0.8}B = 2.8136$
$\mathbf{D}^{0.9}[f(x)]$	$m_{0.9} = 11.7047$	$\theta_{0.9} = 1.4856$	$\triangle PA_{0.9}B = 2.7339$
$\mathbf{D}^{1.0}[f(x)]$	$m_{1.0} = 12.0000$	$\theta_{1.0} = 1.4877$	$\triangle PA_{1.0}B = 2.6667$

Table 4.2

Fractional order derivatives	Fractional derivative values at $x = 2$ $m = \tan\theta$	$\theta = \tan^{-1}m$ (in radian)	Area of triangle ($\triangle$)
$\mathbf{D}^{0.1}[g(x)]$	$m_{0.1} = 25.787$	$\theta_{0.1} = 1.5320$	$\triangle PA_{0.1}B = 11.1684$
$\mathbf{D}^{0.2}[g(x)]$	$m_{0.2} = 27.642$	$\theta_{0.2} = 1.5346$	$\triangle PA_{0.2}B = 10.4188$
$\mathbf{D}^{0.3}[g(x)]$	$m_{0.3} = 29.561$	$\theta_{0.3} = 1.5370$	$\triangle PA_{0.3}B = 9.7427$
$\mathbf{D}^{0.4}[g(x)]$	$m_{0.4} = 31.535$	$\theta_{0.4} = 1.5391$	$\triangle PA_{0.4}B = 9.1328$
$\mathbf{D}^{0.5}[g(x)]$	$m_{0.5} = 33.557$	$\theta_{0.5} = 1.5410$	$\triangle PA_{0.5}B = 8.5825$
$\mathbf{D}^{0.6}[g(x)]$	$m_{0.6} = 35.617$	$\theta_{0.6} = 1.5427$	$\triangle PA_{0.6}B = 8.0860$
$\mathbf{D}^{0.7}[g(x)]$	$m_{0.7} = 37.705$	$\theta_{0.7} = 1.5443$	$\triangle PA_{0.7}B = 7.6383$
$\mathbf{D}^{0.8}[g(x)]$	$m_{0.8} = 39.807$	$\theta_{0.8} = 1.5457$	$\triangle PA_{0.8}B = 7.2349$
$\mathbf{D}^{0.9}[g(x)]$	$m_{0.9} = 41.911$	$\theta_{0.9} = 1.5469$	$\triangle PA_{0.9}B = 6.8718$
$\mathbf{D}^{1.0}[g(x)]$	$m_{1.0} = 44.000$	$\theta_{1.0} = 1.5481$	$\triangle PA_{1.0}B = 6.5455$

Consider the function $f(x)$=x^3at $P(2, 8)$ we have $\mathrm{D}^{1.0}$[f(x)]=12.00. Now with the tangent line $l_{1.0}$ drawn at $P(2, 8)$ which passes through the X-axes at A_1 and with the perpendicular line from $P(2, 8)$ to X-axes at $B(2, 0)$ we have an area ($\triangle$) enclosed by triangle PA_1B=2.6667 ($\triangle PA_1B = 2.6667$). Similarly all the triangles are formed by using fractional derivative values $m_{0.1}, m_{0.2}, \ldots, m_{0.9}$ with tangent line $l_{0.1}, l_{0.2}, \ldots, l_{0.9}$ passing through point $P(2, 8)$. The areas of triangles are computed and the related results are shown in table 4.1.

Similarly, the areas of triangles ($\triangle$) for the function $g(x) = x^4 + x^3$ at $P(2, 24)$ are computed and the results are shown in table 4.2.

Figure.1 and Figure.2 show the graphs of the functions $f(x)$ and $g(x)$ with triangles formed by fractional derivatives of order 0.2, 0.4, 0.6 and 0.8.

From Tables 4.1 and 4.2 and from the graphs of the functions $f(x)$ and $g(x)$, it is observed that if the value of fractional order derivative increases, then the area of triangle decreases, and if the value of fractional order derivative decreases, then the area of triangle increases. Hence fractional order derivative values and areas of triangles are inversely proportional. Further,

$$D^{\alpha}[f(x)] \propto \frac{1}{\triangle}$$
$$D^{\alpha}[g(x)] \propto \frac{1}{\triangle}$$

infer

$$D^{\alpha}[f(x)].\triangle = D^{\alpha}[g(x)].\triangle = \text{ constant.}$$

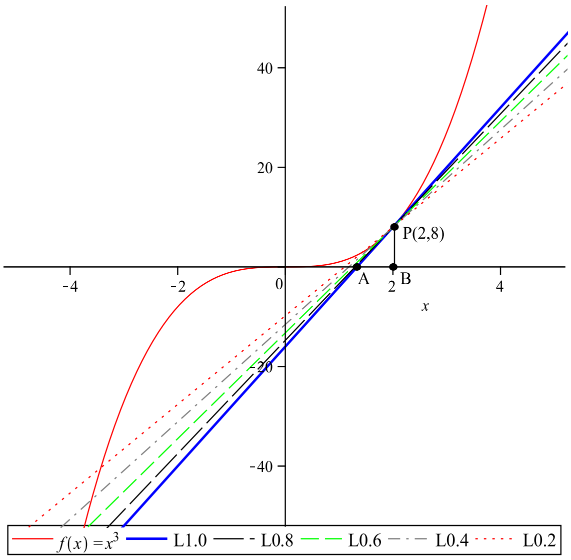

Figure 1: Graph of the function $f(x) = x^3$ with triangles formed with fractional order derivatives.

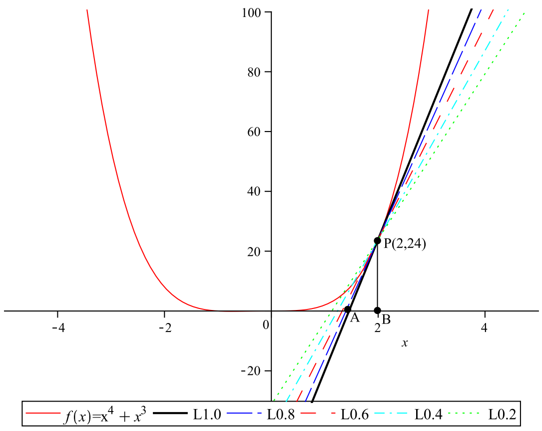

Figure 2: graph of function $g(x) = x^4 + x^3$ with triangles formed with fractional order derivatives

We conclude that the product of fractional order derivative with the correspondent area is constant, so the fractional derivative produces the change in the area of the triangle enclosed by the tangent line at particular point and vertical line passing through this point and above X-axes with respect to fractional gradient line.

The change of area is a physical property, therefore fractional derivatives can be used to measure the changes in temperature, pressure, gradient, divergence and curl, etc.

5 Application in physical quantity divergence

Area and divergence are physical quantities. Fractional order derivative produces the change in area of a triangle mentioned in section 4. In this section we will show that fractional order derivative produces the changes in divergence of vector field.

5.1 The divergence

In physical terms, the divergence of a three dimensional vector field is the extent to which the vector field flow behaves like a source or a sink at a given point.

Let $x,\ y,\ z$ be a system of Cartesian coordinates on 3-dimensional space and let $i,\ j,\ k$ be the corresponding basis of unit vector [3].

The divergence of continuous differentiable vector field $F = F_1 \overrightarrow{i} + F_2 \overrightarrow{j} + F_3 \overrightarrow{k}$ is defined to be the scalar-valued function given by

$$Div\ F = \nabla . F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \tag{5.1}$$

Thus the divergence at a point measures how much the vector field F “spreads out” at the point. A positive divergence means that vector field has a net expansion from the point. A negative divergence means it has a net contraction into the point.

5.2 Fractional divergence

Fractional divergence of a vector filed F can be defined by using the concepts of fractional order derivatives:

$$\nabla^{\alpha} . F = \frac{\partial^{\alpha} F_1}{\partial x^{\alpha}} + \frac{\partial^{\alpha} F_2}{\partial y^{\alpha}} + \frac{\partial^{\alpha} F_3}{\partial z^{\alpha}}, \tag{5.2}$$

where α is the order of derivative and $0 < \alpha < 1$.

5.3 Comparison of two model of divergence

One might consider, e.g., $K(x, y, z) = x^4 + y^3 + z^2$, two points $A(1, 1, 1)$ and $B(4, 1, 2)$ and compare the divergence of gradient K with fractional divergence of gradient K at these points.

5.3.1 Divergence of $\nabla \mathbf{K}(\mathbf{x}, \mathbf{y}, \mathbf{z})$

Let $\nabla K = F = 4x^3 \overrightarrow{i} + 3y^2 \overrightarrow{j} + 2z \overrightarrow{k}$

$$\begin{aligned} Div\ F = \nabla . F &= \frac{\partial}{\partial x}\left(4x^3\right) + \frac{\partial}{\partial y}\left(3y^2\right) + \frac{\partial}{\partial z}\left(2z\right) \\ &= 12x^2 + 6y + 2. \end{aligned} \tag{5.3}$$

Therefore at $A(1, 1, 1)$ we have

$$\nabla . F(1, 1, 1) = 20$$
$$|F(1, 1, 1)| = \sqrt{29} = 5.3852$$

Table 5.1

fractional divergence at $B(4,1,2)$	fractional divergence at $A(1,1,1)$
$\nabla^{0.1}.F =$ 259.49	$\nabla^{0.1}.F = 9.8918$
$\nabla^{0.2}.F =$ 255.30	$\nabla^{0.2}.F = 10.8390$
$\nabla^{0.3}.F =$ 250.44	$\nabla^{0.3}.F = 11.8399$
$\nabla^{0.4}.F =$ 244.93	$\nabla^{0.4}.F = 12.8920$
$\nabla^{0.5}.F =$ 238.30	$\nabla^{0.5}.F = 13.9919$
$\nabla^{0.6}.F =$ 232.07	$\nabla^{0.6}.F = 15.1348$
$\nabla^{0.7}.F =$ 224.79	$\nabla^{0.7}.F = 16.3149$
$\nabla^{0.8}.F =$ 216.98	$\nabla^{0.8}.F = 17.5250$
$\nabla^{0.9}.F =$ 208.70	$\nabla^{0.9}.F = 18.7567$

As it can be observed, $\nabla.F > |F|$. Similarly, at $B(4,1,2)$, we have

$$\nabla.F(4,1,2) = 200$$
$$|F(4,1,2)| = \sqrt{65561} = 256.0488$$

and hence $\nabla.F < |F|$.

5.3.2 Fractional divergence of $\nabla\mathbf{K}(x,y,z)$

Using the formula

$$D^{\alpha}\left[x^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha},$$

the fractional divergence values are computed for $\alpha = 0.1, 0.2, \ldots, 0.9$ at two points $A(1,1,1)$ and $B(4,1,2)$. As shown in table 5.1, it is observed that:

1. When $\nabla.F < |F|$ at $B(4,1,2)$ the amount fractional divergence of vector field F is decreasing from $\nabla^{0.1}.F$ to $\nabla^{0.9}.F$, Fractional divergence of vector field F at $\alpha = 0.1$ is very high than the divergence of F.

2. When $\nabla.F > |F|$ at $A(1,1,1)$ the amount fractional divergence of vector field F is increasing from $\nabla^{0.1}.F$ to $\nabla^{0.9}.F$, Fractional divergence of vector field F at $\alpha = 0.1$ is very low than the divergence of F.

It is suggested that to obtain the higher amount of vector field spread out, we can use the fractional divergence $\nabla^{0.1}.F$, rather than $\nabla.F$. Thus geometric and physical interpretation of fractional order derivative of a polynomial function play important role for measuring the changes in physical quantities.

6 Critical point of fractional derivatives for polynomial functions

According to the table 6.1, for values taken by fractional derivatives $D^{\alpha}[x^{\beta}]$ at different values of $\beta = 2, 3, \ldots, 10$, it can be seen that there are two different results for $x = \beta$ and $x \neq \beta$. Namely, there is a monotonically increasing or decreasing trend for the case $x \neq \beta$. But it is somehow different for the case $x = \beta$, and we will prove that in this case $\alpha = 0.5$ is a critical point for $D^{\alpha}[x^{\beta}]$.

We set

$$D(x, \beta, \alpha) = D^{\alpha}[x^{\beta}].$$

For $x = \beta$, we have

$$D(\beta, \beta, \alpha) = D(\beta, \alpha) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} \beta^{\beta-\alpha},$$

and hence

$$\frac{\partial D}{\partial \alpha} = \frac{\Gamma\left(\beta+1\right) \beta^{(\beta-\alpha)} \left[\psi\left(\beta+1-\alpha\right) - \ln\left(\beta\right)\right]}{\Gamma\left(\beta+1-\alpha\right)} = 0, \tag{6.1}$$

where

$$\psi(x) = \frac{d}{dx} \ln\left(\Gamma(x)\right) = \frac{\frac{d}{dx}\Gamma(x)}{\Gamma(x)} .$$

The numerical solutions of (6.1) for different values of $\beta = 2, 3, \ldots, 10$ give the critical point $\alpha = 0.5$ in the interval $0 < \alpha < 1$.

Hence, according with results of table 6.1 for $D^{\alpha}\left[f(x)\right] = D^{\alpha}[x^{\beta}]$ and also from the produced triangles of Section 4, we conclude that:

1. If $x > \beta$, then the value of fractional derivative (m) decrease and the area of triangle $(\triangle)$ increase.

2. If $x < \beta$, then the value of fractional derivative (m) increase and the area of triangle $(\triangle)$ decrease.

3. If $x = \beta$ then the value of fractional derivative increase from $\alpha = 0.1$ to $\alpha = 0.5$ and decrease from $\alpha = 0.5$ to $\alpha = 1.0$, and conversely the area of triangle primarily decrease and then increase.

Table 6.1 fractional derivative values for polynomial function $f(x) = x^\beta$

x = 1					
$\mathbf{D}^{\alpha}[f(x)]$	β= 2	β= 4	β= 6	β= 8	β= 10
$\mathbf{D}^{0.1}[f(x)]$	1.0945	1.1612	1.2050	1.2380	1.2645
$\mathbf{D}^{0.2}[f(x)]$	1.1930	1.3455	1.4498	1.5308	1.5975
$\mathbf{D}^{0.3}[f(x)]$	1.2948	1.5553	1.7416	1.8905	2.0162
$\mathbf{D}^{0.4}[f(x)]$	1.3990	1.7935	2.0888	2.3320	2.5421
$\mathbf{D}^{0.5}[f(x)]$	1.5045	2.0633	2.5010	2.8729	3.2020
$\mathbf{D}^{0.6}[f(x)]$	1.6101	2.3678	2.9896	3.5350	4.0293
$\mathbf{D}^{0.7}[f(x)]$	1.7142	2.7102	3.5677	4.3442	5.0651
$\mathbf{D}^{0.8}[f(x)]$	1.8152	3.0941	4.2501	5.3317	6.3607
$\mathbf{D}^{0.9}[f(x)]$	1.9112	3.5229	5.0543	6.5353	7.9796
$\mathbf{D}^{1.0}[f(x)]$	2.0000	4.0000	6.0000	8.0000	10.0000
x = 2					
$\mathbf{D}^{\alpha}[f(x)]$	β= 2	β= 4	β= 6	β= 8	β= 10
$\mathbf{D}^{0.1}[f(x)]$	4.0847	17.3358	71.9575	295.6975	1208.1614
$\mathbf{D}^{0.2}[f(x)]$	4.1542	18.7406	80.7783	341.1449	1424.0745
$\mathbf{D}^{0.3}[f(x)]$	4.2067	20.2123	90.5368	393.1041	1676.9461
$\mathbf{D}^{0.4}[f(x)]$	4.2409	21.7481	101.3110	452.4257	1972.7864
$\mathbf{D}^{0.5}[f(x)]$	4.2554	23.3438	113.1822	520.0576	2318.5229
$\mathbf{D}^{0.6}[f(x)]$	4.2490	24.9944	126.2342	597.0537	2722.1293
$\mathbf{D}^{0.7}[f(x)]$	4.2209	26.6935	140.5535	684.5834	3192.7713
$\mathbf{D}^{0.8}[f(x)]$	4.1703	28.4335	156.2282	783.9408	3740.9688
$\mathbf{D}^{0.9}[f(x)]$	4.0967	30.2058	173.3473	896.5550	4378.7789
$\mathbf{D}^{1.0}[f(x)]$	4.0000	32.0000	192.0000	1024.0000	5120.0000
x = 3					
$\mathbf{D}^{\alpha}[f(x)]$	β= 2	β= 4	β= 6	β= 8	β= 10
$\mathbf{D}^{0.1}[f(x)]$	8.8255	84.2750	787.0716	7277.2721	66900.3561
$\mathbf{D}^{0.2}[f(x)]$	8.6188	87.4841	848.4451	8062.1480	75722.8647
$\mathbf{D}^{0.3}[f(x)]$	8.3810	90.6054	913.1566	8920.9332	85625.7365
$\mathbf{D}^{0.4}[f(x)]$	8.1134	93.6160	981.2232	9859.1806	96728.8799
$\mathbf{D}^{0.5}[f(x)]$	7.8176	96.4920	1052.6402	10882.6800	109163.7247
$\mathbf{D}^{0.6}[f(x)]$	7.4958	99.2093	1127.3788	11997.4435	123074.0782
$\mathbf{D}^{0.7}[f(x)]$	7.1503	101.7433	1205.3840	13209.6875	138617.0081
$\mathbf{D}^{0.8}[f(x)]$	6.7838	104.0694	1286.5717	14525.8090	155963.7496
$\mathbf{D}^{0.9}[f(x)]$	6.3993	106.1629	1370.8266	15952.3573	175300.6294
$\mathbf{D}^{1.0}[f(x)]$	6.0000	108.0000	1458.0000	17496.0000	196830.0000
x = 4					
$\mathbf{D}^{\alpha}[f(x)]$	β= 2	β= 4	β= 6	β= 8	β= 10
$\mathbf{D}^{0.1}[f(x)]$	15.2448	258.7973	4296.8758	70629.2560	1154308.5756
$\mathbf{D}^{0.2}[f(x)]$	14.4656	261.0336	4500.5793	76027.8852	1269482.3128
$\mathbf{D}^{0.3}[f(x)]$	13.6676	262.6803	4706.4785	81740.7390	1394793.9826
$\mathbf{D}^{0.4}[f(x)]$	12.8559	263.7117	4913.8822	87775.8861	1530974.7569
$\mathbf{D}^{0.5}[f(x)]$	12.0360	264.1052	5122.0403	94140.4744	1678789.8834
$\mathbf{D}^{0.6}[f(x)]$	11.2133	263.8422	5330.1453	100840.5866	1839038.0537
$\mathbf{D}^{0.7}[f(x)]$	10.3931	262.9080	5537.3340	107881.0886	2012550.4280
$\mathbf{D}^{0.8}[f(x)]$	9.5807	261.2924	5742.6905	115265.4717	2200189.2803
$\mathbf{D}^{0.9}[f(x)]$	8.7814	258.9899	5945.2493	122995.6913	2402846.2271
$\mathbf{D}^{1.0}[f(x)]$	8.0000	256.0000	6144.0000	131072.0000	2621440.0000
x = 5					
$\mathbf{D}^{\alpha}[f(x)]$	β= 2	β= 4	β= 6	β= 8	β= 10
$\mathbf{D}^{0.1}[f(x)]$	23.2943	617.8865	16029.5703	411693.2382	10513106.1845
$\mathbf{D}^{0.2}[f(x)]$	21.6160	609.4730	16418.9920	433382.1419	11306932.0407
$\mathbf{D}^{0.3}[f(x)]$	19.9728	599.7836	16791.2548	455664.9881	12148906.5448
$\mathbf{D}^{0.4}[f(x)]$	18.3722	588.8510	17144.3417	478510.3337	13040797.6114
$\mathbf{D}^{0.5}[f(x)]$	16.8209	576.7160	17476.2426	501881.8381	13984323.6623
$\mathbf{D}^{0.6}[f(x)]$	15.3252	563.4278	17784.9692	525738.1101	14981139.1567
$\mathbf{D}^{0.7}[f(x)]$	13.8908	549.0436	18068.5703	550032.5803	16032819.0911
$\mathbf{D}^{0.8}[f(x)]$	12.5225	533.6283	18325.1474	574713.4032	17140842.4877
$\mathbf{D}^{0.9}[f(x)]$	11.2244	517.2541	18552.8716	599723.3938	18306574.9028
$\mathbf{D}^{1.0}[f(x)]$	10.0000	500.0000	18750.0000	625000.0000	19531250.0000

Table 6.1(continue)fractional derivative values for polynomial function $f(x) = x^{\beta}$

		$\mathbf{x} = 6$			
$\mathbf{D}^{\alpha}[f(x)]$	β= 2	β= 4	β= 6	β= 8	β= 10
$\mathbf{D}^{0.1}[f(x)]$	32.9378	1258.1012	46999.2829	1738223.3428	63918325.1678
$\mathbf{D}^{0.2}[f(x)]$	30.0124	1218.5496	47271.3188	1796737.9082	67502676.5132
$\mathbf{D}^{0.3}[f(x)]$	27.2300	1177.5117	47469.6746	1854988.6412	71218902.6845
$\mathbf{D}^{0.4}[f(x)]$	24.5952	1135.1621	47592.2011	1912796.5993	75066145.6135
$\mathbf{D}^{0.5}[f(x)]$	22.1116	1091.6825	47637.0564	1969975.5011	79042979.8590
$\mathbf{D}^{0.6}[f(x)]$	19.7816	1047.2600	47602.7285	2026332.3617	83147376.5421
$\mathbf{D}^{0.7}[f(x)]$	17.6061	1002.0859	47488.0562	2081668.2156	87376668.2004
$\mathbf{D}^{0.8}[f(x)]$	15.5850	956.3543	47292.2477	2135778.9274	91727514.9076
$\mathbf{D}^{0.9}[f(x)]$	13.7171	910.2607	47014.8974	2188456.0887	96195872.0294
$\mathbf{D}^{1.0}[f(x)]$	12.0000	864.0000	46656.0000	2239488.0000	100776960.0000
		$\mathbf{x} = 7$			
$\mathbf{D}^{\alpha}[f(x)]$	β= 2	β= 4	β= 6	β= 8	β= 10
$D^{0.1}[f(x)]$	44.1462	2295.1340	116701.7291	5874693.5375	294034712.2946
$D^{0.2}[f(x)]$	39.6100	2188.9760	115581.7054	5979566.3593	305773279.7374
$D^{0.3}[f(x)]$	35.3880	2082.8996	114291.2429	6078991.4810	317672146.3569
$D^{0.4}[f(x)]$	31.4749	1977.2716	112833.4359	6172546.8110	329710894.3358
$D^{0.5}[f(x)]$	27.8638	1872.4496	111212.1561	6259818.5896	341867492.0140
$D^{0.6}[f(x)]$	24.5463	1768.7792	109432.0438	6340403.8889	354118302.3061
$D^{0.7}[f(x)]$	21.5126	1666.5923	107498.4935	6413913.1578	366438101.1276
$D^{0.8}[f(x)]$	18.7518	1566.2048	105417.6332	6479972.7951	378800106.3946
$D^{0.9}[f(x)]$	16.2519	1467.9147	103196.2985	6538227.7313	391176018.1099
$D^{1.0}[f(x)]$	14.0000	1372.0000	100842.0000	6588344.0000	403536070.0000
		$x = 8$			
$D^{\alpha}[f(x)]$	β= 2	β= 4	β= 6	β= 8	β= 10
$D^{0.1}[f(x)]$	56.8955	3863.4620	256584.1222	16870253.0521	1102856175.0096
$D^{0.2}[f(x)]$	50.3722	3635.8871	250750.8366	16943646.2770	1131672107.5556
$D^{0.3}[f(x)]$	44.4061	3413.8036	244662.2996	16996892.4551	1160114948.9428
$D^{0.4}[f(x)]$	38.9719	3197.6973	238337.6847	17029550.6740	1188108186.5610
$D^{0.5}[f(x)]$	34.0431	2988.0093	231797.0843	17041246.1582	1215573719.7697
$D^{0.6}[f(x)]$	29.5921	2785.1349	225061.4041	17031673.8254	1242432133.1620
$D^{0.7}[f(x)]$	25.5908	2589.4218	218152.2502	17000601.5354	1268602990.5955
$D^{0.8}[f(x)]$	22.0107	2401.1693	211091.8110	16947872.9956	1294005149.1848
$D^{0.9}[f(x)]$	18.8233	2220.6282	203902.7343	16873410.2899	1318557092.2480
$D^{1.0}[f(x)]$	16.0000	2048.0000	196608.0000	16777216.0000	1342177280.0000
		$x = 9$			
$D^{\alpha}[f(x)]$	β= 2	β= 4	β= 6	β= 8	β= 10
$D^{0.1}[f(x)]$	71.1652	6116.0564	514078.7598	42778595.7525	3539393519.8682
$D^{0.2}[f(x)]$	62.2681	5688.3980	496508.8808	42461619.2210	3589346058.9210
$D^{0.3}[f(x)]$	54.2503	5278.4070	478780.4751	42096302.2859	3636474033.2261
$D^{0.4}[f(x)]$	47.0539	4886.3709	460942.5942	41683325.5086	3680613407.9176
$D^{0.5}[f(x)]$	40.6217	4512.4850	443043.9825	41223538.5607	3721604905.3594
$D^{0.6}[f(x)]$	34.8970	4156.8544	425132.8315	40717958.6896	3759294818.2299
$D^{0.7}[f(x)]$	29.8251	3819.4965	407256.5388	40167768.2083	3793535823.7890
$D^{0.8}[f(x)]$	25.3523	3500.3451	389461.4734	39574311.0023	3824187794.3588
$D^{0.9}[f(x)]$	21.4271	3199.2536	371792.7477	38939088.0550	3851118598.8500
$D^{1.0}[f(x)]$	18.0000	2916.0000	354294.0000	38263752.0000	3874204890.0000
		$x = 10$			
$D^{\alpha}[f(x)]$	β= 2	β= 4	β=6	β= 8	β= 10
$D^{0.1}[f(x)]$	86.9375	9224.1359	957191.5472	98335583.6467	10044492711.6096
$D^{0.2}[f(x)]$	75.2712	8489.2328	914788.0152	96583953.3364	10079494202.5418
$D^{0.3}[f(x)]$	64.8918	7794.8109	872879.1654	94749434.5094	10104809937.0167
$D^{0.4}[f(x)]$	55.6939	7140.2496	831550.8061	92836613.1221	10120270325.8127
$D^{0.5}[f(x)]$	47.5766	6524.7968	790884.4567	90850317.0755	10125731934.1131
$D^{0.6}[f(x)]$	40.4435	5947.5798	750957.0422	88795596.2014	10121078594.3850
$D^{0.7}[f(x)]$	34.2032	5407.6159	711840.6187	86677700.9122	10106222414.9483
$D^{0.8}[f(x)]$	28.7691	4903.8235	673602.1325	84502059.6336	10081104675.2658
$D^{0.9}[f(x)]$	24.0601	4435.0334	636303.2126	82274255.1495	10045696599.4469
$D^{1.0}[f(x)]$	20.0000	4000.0000	600000.0000	80000000.0000	10000000000.0000

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Author's address:

Mohammad Hossein Tavassoli
Department of Mathematics and Computer Sciences,
Khomein Branch, Islamic Azad University, Khomein, Iran.
E-mail: Tavassoli.1359@gmail.com

Abbas Tavassoli
Department of Mathematics and Computer Sciences,

Payam Institute of Higher Education, Golpayegan, Iran.
E-mail: abs.tavassoli@yahoo.com

Mohammad Reza Ostad Rahimi
Golpayegan Branch, Islamic Azad University, Golpayegan, Iran.
E-mail: mrostadrahimi@yahoo.com